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A systematic derivation of the leading-order equations for extensional flows in slender geometries

By J. N. DEWYNNE[1], J. R. OCKENDON[2] AND P. WILMOTT[2]

[1] Faculty of Mathematical Studies, University of Southampton, Southampton, SO9 5NH, UK

[2] Mathematical Institute, Oxford University, Oxford, OX1 3LB, UK

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We consider the extensional flow and twist of a finite, slender, nearly straight, Newtonian viscous fibre when its ends are drawn apart at prescribed velocity. The initial cross-section of the fibre may be arbitrary and may vary gradually in the axial direction. We derive the leading-order equations for the fibre's free surface and its flow velocity from a regular perturbation expansion of the full Stokes' flow problem in powers of the aspect ratio. In order to obtain these equations systematically, it is necessary to consider terms beyond the leading order in the perturbation expansion, because those obtainable from the leading-order terms give an indeterminate set of equations. Our results are a pair of well-known hyperbolic equations for the area and axial velocity, together with (i) straightness of the line of centres of mass of the cross-section and (ii) a new hyperbolic evolution equation for the twist of the cross-section. It is only through this hyperbolic equation that history effects are manifest.

1. Introduction

The problem of determining the evolution of the cross-sectional area, $A(x,t)$, and the axial velocity, $u(x,t)$, of a slender, nearly straight, axisymmetric, Newtonian viscous fibre can be described by the equations

$$A_t+(uA)_x=0, \quad (3\mu A u_x)_x=0, \tag{1.1 a, b}$$

where the axis of symmetry is along the x-axis, μ is the dynamic viscosity of the fibre and 3μ its so-called Trouton viscosity. These equations are well known in the context of molten glass draw-down (Geyling & Homsy 1980), glass-fibre tapering (Dewynne, Ockendon & Wilmott 1989; Geyling 1976), and polymer-fibre production (Pearson & Matovich 1969). The various additional effects of surface-tension, coupled heat flow, gravity, small initial asymmetry, aerodynamic forces, more complicated rheology and inertia have been considered by Geyling (1976), Geyling & Homsy (1980), Kasé (1974), Person & Matovich (1969), Entov & Yarin (1984), Myers (1989), Schultz & Davis (1982), Beris & Liu (1988), Bechtel *et al.* (1988) and Shah & Pearson (1972).

Equations (1.1) can be derived from the assumption of purely extensional flow and the application of one-dimensional mass and momentum conservation (to yield (1.1*a*) and (1.1*b*) respectively). Alternatively they can be derived systematically by the use of regular asymptotic expansions of the full Stokes' equations for the physical problem, where they emerge as the leading-order equations (Dewynne *et al.* 1989; Wilmott 1989). Using the systematic approach it is, however, necessary to consider terms of order higher than leading-order in the asymptotic series to obtain the momentum equation (1.1*b*) (Schultz & Davis 1982, Entov & Yarin 1984). It is also necessary to consider these higher-order terms to deduce that the fibre's centreline will remain straight if it is initially straight. This matter has been discussed in the two-dimensional case, in which the centreline is not initially straight, in Buckmaster & Nachman (1978), Buckmaster, Nachman & Ting (1975) and Wilmott (1989). It is interesting that while the model seems to be more difficult in the 'fully nonlinear' case, its derivation is actually quicker than the 'nearly straight' case. We will return to this in the conclusion.

The situation is somewhat analogous to that in the theory of elastostatics with geometric nonlinearity. There, the Kirchhoff equations of elastica theory can be derived either by making certain assumptions about the stress distribution (Love 1927) or by a systematic asymptotic analysis of the equations of three-dimensional elasticity (Parker 1984). The latter approach, however, again requires consideration of first-order terms as well as leading-order ones. This situation seems to arise whenever a thin-layer problem in continuum mechanics is treated by approximating a three-space-dimensional elliptic problem by a lower-space-dimensional parabolic problem and the driving mechanism is only applied at the ends of the thin layer. Then the fact that the leading-order motion along the length of the layer is some non-trivial eigensolution of a parabolic differential equation only permits its precise determination via the Fredholm alternative applied to the first-order non-homogeneous parabolic differential equation. Examples where this happens are shallow-water theory and theories of plates and shells as well as those mentioned above, but not classical boundary-layer or lubrication theory.

In this paper we consider the problem of simultaneously stretching and twisting a finite, slender, viscous fibre of arbitrary cross-sectional shape that may vary gradually in the axial direction. In order to generalize (1.1), to determine the evolution of the cross-sectional shape, it is no longer enough to assume pure extensional flow and use one-dimensional conservation laws. Instead, we obtain the equations describing the problem to leading-order based on a systematic study of asymptotic expansions of the full Stokes' flow equations for the problem. Although the problem considered in this paper has applications to the draw-down of glass melts with arbitrary cross-sections and the tapering of geometrically complicated optical fibre couplers, we will ignore many physical effects relevant to these problems (such as surface tension, gravity and coupled heat flow). We do so in order to highlight the mathematically interesting feature of the problem, namely, that in order to obtain a well-posed set of equations for the leading-order problem we have to consider higher-order terms. Specifically, we will derive the leading-order equations for the axial momentum, for the displacement of the centreline and for the transmission of the shear stress from consideration of higher-order terms of the asymptotic expansion of the Stokes' flow problem.

2. The scaled equations

Before we start, we note that our field equations and free boundary conditions are all invariant under rigid-body motion. The specification of the positions of the ends of the fibre gives uniqueness, but any rigid-body motion of a line joining these ends will be automatically imposed on the solution.

The equations of Stokes' flow are

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{q} = 0, \tag{2.1}$$

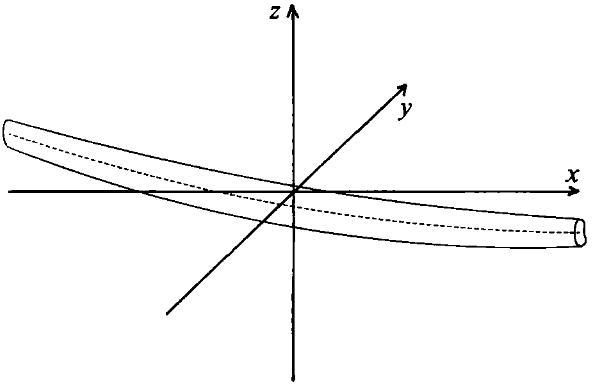

FIGURE 1. The coordinate system.

where $\boldsymbol{\sigma}$ is the stress tensor with components

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

p is the pressure, μ the dynamic viscosity and $\boldsymbol{q} = (u, v, w)$ is the fluid velocity vector. As in figure 1, the x-axis and direction of u are approximately parallel to the fibre, in a sense to be made more precise later (see (4.10)). We ignore gravity and other body forces in (2.1).

The free surface of the fibre is described by an equation of the form

$$G(x, y, z, t) = 0.$$

On the free surface of the fibre we have the kinematic condition

$$\left(\frac{\partial}{\partial t} + (\boldsymbol{q}\cdot\boldsymbol{\nabla})\right)G(x, y, z, t) = 0 \quad \text{on} \quad G(x, y, z, t) = 0, \tag{2.2}$$

and the dynamic conditions are, in the absence of surface tension, the zero-stress conditions

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad G(x, y, z, t) = 0, \tag{2.3}$$

where $\boldsymbol{n}$ is the unit normal to the free surface. Equations (2.1)–(2.3) are invariant under rigid-body changes to $\boldsymbol{q}$ and G. We obtain a unique solution when we prescribe $\boldsymbol{q}$ at the ends of the fibre; thinking of a situation where the ends are attached to rigid planes perpendicular to the x-axis we put

$$\boldsymbol{q} = \boldsymbol{q}_1^*(t) \quad \text{at} \quad x = s_1(t), \quad \boldsymbol{q} = \boldsymbol{q}_2^*(t) \quad \text{at} \quad x = s_2(t),$$

where $x = s_i(t)$ are the ends of the fibre. Note that if $\boldsymbol{q}_i^*(t) = (u_i, v_i, w_i)$ then $u_i = \dot{s}_i$, and v_i and w_i will describe a twist applied at the ends. We also prescribe some initial form for the fibre shape $G(x, y, z, 0) = 0$.

Letting ϵ be the aspect ratio of the fibre, we non-dimensionalize and scale the problem with

$$\boldsymbol{q} = U(\bar{u}, \epsilon\bar{v}, \epsilon\bar{w}), \quad \boldsymbol{x} = L(\bar{x}, \epsilon\bar{y}, \epsilon\bar{z}), \quad s_i(t) = L\bar{s}_i(\bar{t}) \quad (i = 1, 2),$$

$$p = \frac{\mu U}{L}\bar{p}, \quad t = \frac{L}{U}\bar{t}, \quad G(x, y, z, t) = \bar{G}(\bar{x}, \bar{y}, \bar{z}, \bar{t}).$$

Here, U is a typical drawing speed, representative of the fibre-end speed difference, $\dot{s}_2-\dot{s}_1$, L is the initial axial length of the fibre and ϵL a typical radial length, and it is assumed that $G(x,y,z,0)$ can be expanded asymptotically in terms of $\bar{x},\bar{y}$ and $\bar{z}$. The slender geometry of the fibre implies that $\epsilon \ll 1$ and our analysis is, therefore, based on expansions of the scaled dependent variables $\bar{u},\bar{v},\bar{w},\bar{\boldsymbol{\sigma}},\bar{p}$ and $\bar{G}$ as regular asymptotic series in even powers of ϵ. We will see in §4, however, that the choice of L/U as the timescale is not always appropriate.

In non-dimensional scaled variables, the stress tensor becomes

$$\bar{\boldsymbol{\sigma}} = \begin{pmatrix} -\bar{p}+2\dfrac{\partial\bar{u}}{\partial\bar{x}} & \dfrac{1}{\epsilon}\dfrac{\partial\bar{u}}{\partial\bar{y}}+\epsilon\dfrac{\partial\bar{v}}{\partial\bar{x}} & \dfrac{1}{\epsilon}\dfrac{\partial\bar{u}}{\partial\bar{z}}+\epsilon\dfrac{\partial\bar{w}}{\partial\bar{x}} \\ \dfrac{1}{\epsilon}\dfrac{\partial\bar{u}}{\partial\bar{y}}+\epsilon\dfrac{\partial\bar{v}}{\partial\bar{x}} & -\bar{p}+2\dfrac{\partial\bar{v}}{\partial\bar{y}} & \dfrac{\partial\bar{v}}{\partial\bar{z}}+\dfrac{\partial\bar{w}}{\partial\bar{y}} \\ \dfrac{1}{\epsilon}\dfrac{\partial\bar{u}}{\partial\bar{z}}+\epsilon\dfrac{\partial\bar{w}}{\partial\bar{x}} & \dfrac{\partial\bar{v}}{\partial\bar{z}}+\dfrac{\partial\bar{w}}{\partial\bar{y}} & -\bar{p}+2\dfrac{\partial\bar{w}}{\partial\bar{z}} \end{pmatrix}, \tag{2.4}$$

and so we expand $\bar{\boldsymbol{\sigma}}$ as

$$\bar{\boldsymbol{\sigma}} = \begin{pmatrix} \bar{\sigma}_{xx0}+\epsilon^2\bar{\sigma}_{xx1} & \epsilon^{-1}\bar{\sigma}_{xy0}+\epsilon\bar{\sigma}_{xy1} & \epsilon^{-1}\bar{\sigma}_{xz0}+\epsilon\bar{\sigma}_{xz1} \\ \epsilon^{-1}\bar{\sigma}_{xy0}+\epsilon\bar{\sigma}_{xy1} & \bar{\sigma}_{yy0}+\epsilon^2\bar{\sigma}_{yy1} & \sigma_{yz0}+\epsilon^2\bar{\sigma}_{yz1} \\ \epsilon^{-1}\bar{\sigma}_{xz0}+\epsilon\sigma_{xz1} & \bar{\sigma}_{yz0}+\epsilon^2\bar{\sigma}_{yz1} & \bar{\sigma}_{zz0}+\epsilon^2\bar{\sigma}_{zz1} \end{pmatrix} + O(\epsilon^3). \tag{2.5}$$

We expand the other dependent variables, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$ and $\bar{G}$ as regular asymptotic series in ϵ^2, so that, for example, $\bar{u} \sim \bar{u}_0+\epsilon^2 u_1+O(\epsilon^4)$.

Substituting into (2.1), and dropping the overbars here and henceforth, gives

$$\frac{\partial\sigma_{xy0}}{\partial y}+\frac{\partial\sigma_{xz0}}{\partial z}+\epsilon^2\left(\frac{\partial\sigma_{xx0}}{\partial x}+\frac{\partial\sigma_{xy1}}{\partial y}+\frac{\partial\sigma_{xz1}}{\partial z}\right) = O(\epsilon^4), \tag{2.6a}$$

$$\frac{\partial\sigma_{xy0}}{\partial x}+\frac{\partial\sigma_{yy0}}{\partial y}+\frac{\partial\sigma_{yz0}}{\partial z}+\epsilon^2\left(\frac{\partial\sigma_{xy1}}{\partial x}+\frac{\partial\sigma_{yy1}}{\partial y}+\frac{\partial\sigma_{yz1}}{\partial z}\right) = O(\epsilon^4), \tag{2.6b}$$

$$\frac{\partial\sigma_{xz0}}{\partial x}+\frac{\partial\sigma_{yz0}}{\partial y}+\frac{\partial\sigma_{zz0}}{\partial z}+\epsilon^2\left(\frac{\partial\sigma_{xz1}}{\partial x}+\frac{\partial\sigma_{yz1}}{\partial y}+\frac{\partial\sigma_{zz1}}{\partial z}\right) = O(\epsilon^4), \tag{2.6c}$$

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}+\epsilon^2\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z}\right) = O(\epsilon^4). \tag{2.6d}$$

Boundary conditions are found by expanding

$$G(x,y,z,t) = G_0(x,y,z,t)+\epsilon^2 G_1(x,y,z,t)+O(\epsilon^4)$$

and linearizing about the leading-order free surface, $G_0(x,y,z,t) = 0$. The kinematic condition, (2.2), which we only need to leading order, becomes

$$\frac{\partial G_0}{\partial t}+u_0\frac{\partial G_0}{\partial x}+v_0\frac{\partial G_0}{\partial y}+w_0\frac{\partial G_0}{\partial z} = 0 \quad \text{on} \quad G_0(x,y,z,t) = 0, \tag{2.7}$$

with $G_0(x,y,z,0)$ prescribed at time $t = 0$.

The zero-stress boundary conditions, (2.3), become

$$\sigma_{xy0}\frac{\partial G_0}{\partial y}+\sigma_{xz0}\frac{\partial G_0}{\partial z}+\epsilon^2\bigg(\sigma_{xx0}\frac{\partial G_0}{\partial x}+\sigma_{xy1}\frac{\partial G_0}{\partial y}+\sigma_{xz1}\frac{\partial G_0}{\partial z}+\sigma_{xy0}\frac{\partial G_1}{\partial y}+\sigma_{xz0}\frac{\partial G_1}{\partial z}$$
$$-\frac{G_1\nabla G_0}{|\nabla G_0|^2}\cdot\nabla\bigg(\sigma_{xy0}\frac{\partial G_0}{\partial y}+\sigma_{xz0}\frac{\partial G_0}{\partial z}\bigg)\bigg)=0, \quad (2.8a)$$

$$\sigma_{xy0}\frac{\partial G_0}{\partial x}+\sigma_{yy0}\frac{\partial G_0}{\partial y}+\sigma_{yz0}\frac{\partial G_0}{\partial z}+\epsilon^2\bigg(\sigma_{xy1}\frac{\partial G_0}{\partial x}+\sigma_{yy1}\frac{\partial G_0}{\partial y}+\sigma_{yz1}\frac{\partial G_0}{\partial z}+\sigma_{xy0}\frac{\partial G_1}{\partial x}+\sigma_{yy0}\frac{\partial G_1}{\partial y}$$
$$+\sigma_{yz0}\frac{\partial G_1}{\partial z}-\frac{G_1\nabla G_0}{|\nabla G_0|^2}\cdot\nabla\bigg(\sigma_{xy0}\frac{\partial G_0}{\partial x}+\sigma_{yy0}\frac{\partial G_0}{\partial y}+\sigma_{yz0}\frac{\partial G_0}{\partial z}\bigg)\bigg)=0, \quad (2.8b)$$

$$\sigma_{xz0}\frac{\partial G_0}{\partial x}+\sigma_{yz0}\frac{\partial G_0}{\partial y}+\sigma_{zz0}\frac{\partial G_0}{\partial z}+\epsilon^2\bigg(\sigma_{xz1}\frac{\partial G_0}{\partial x}+\sigma_{yz1}\frac{\partial G_0}{\partial y}+\sigma_{zz1}\frac{\partial G_0}{\partial z}+\sigma_{xz0}\frac{\partial G_1}{\partial x}+\sigma_{yz0}\frac{\partial G_1}{\partial y}$$
$$+\sigma_{zz0}\frac{\partial G_1}{\partial z}-\frac{G_1\nabla G_0}{|\nabla G_0|^2}\cdot\nabla\bigg(\sigma_{xz0}\frac{\partial G_0}{\partial x}+\sigma_{yz0}\frac{\partial G_0}{\partial y}+\sigma_{zz0}\frac{\partial G_0}{\partial z}\bigg)\bigg)=0 \quad (2.8c)$$

on $G_0(x,y,z,t)=0$.

3. The leading-order problem

From (2.6*a*) and (2.8*a*) we find, on equating like powers of ϵ^2, that

$$\frac{\partial\sigma_{xy0}}{\partial y}+\frac{\partial\sigma_{xz0}}{\partial z}=0,$$

$$\sigma_{xy0}\frac{\partial G_0}{\partial y}+\sigma_{xz0}\frac{\partial G_0}{\partial z}=0 \quad \text{on} \quad G_0(x,y,z,t)=0.$$

Observing from (2.4) and (2.5) that $\sigma_{xy0}=\partial u_0/\partial y$ and $\sigma_{xz0}=\partial u_0/\partial z$, we find that u_0 satisfies

$$\hat{\nabla}^2u_0=0, \quad \frac{\partial u_0}{\partial\hat{n}_0}=0 \quad \text{on} \quad G_0(x,y,z,t)=0, \tag{3.1}$$

where, here and henceforth, $\hat{\nabla}^2=\partial^2/\partial y^2+\partial^2/\partial z^2$ is the transverse Laplacian, and $\partial u_0/\partial\hat{n}_0$ is the transverse normal derivative with respect to the curve $G_0(x,y,z,t)=0$, with x and t held fixed. (In general, in what follows a hatted variable denotes the two-dimensional, transverse components of the unhatted variable.) Thus, u_0 is an eigensolution of (3.1),

$$u_0=u_0(x,t), \tag{3.2}$$

and $\sigma_{xy0}=\sigma_{xz0}=0$ identically.

From the $O(1)$ terms in (2.6*b–d*), we then have

$$\hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}_0=\mathbf{0}, \quad \hat{\nabla}\cdot\hat{\boldsymbol{q}}_0=-\frac{\partial u_0}{\partial x}(x,t), \tag{3.3a, b}$$

where $\hat{\boldsymbol{\sigma}}_0$, $\hat{\boldsymbol{q}}_0$ and $\hat{\nabla}$ are the leading-order transverse tensor, transverse velocity vector and transverse gradient operator, respectively;

$$\hat{\boldsymbol{\sigma}}_0=\begin{pmatrix}\sigma_{yy0} & \sigma_{yz0}\\ \sigma_{yz0} & \sigma_{zz0}\end{pmatrix}, \quad \hat{\boldsymbol{q}}_0=\begin{pmatrix}v_0\\ w_0\end{pmatrix}, \quad \hat{\nabla}=\begin{pmatrix}\partial/\partial y\\ \partial/\partial z\end{pmatrix}.$$

The leading-order dynamic conditions are, from (2.8b,c) and (3.2),

$$\hat{\boldsymbol{\sigma}}_0 \cdot \hat{\boldsymbol{n}}_0 = \mathbf{0} \quad \text{on} \quad G_0(x,y,z,t) = 0 \tag{3.4}$$

where $\hat{\boldsymbol{n}}_0$ is the transverse normal to the surface $G_0(x,y,z,t)=0$ (with x and t regarded as fixed), defined by

$$\hat{\boldsymbol{n}}_0 = \hat{\boldsymbol{\nabla}} G_0 / |\hat{\boldsymbol{\nabla}} G_0|. \tag{3.5}$$

We can introduce an Airy stress function $\mathfrak{A}(y,z;x,t)$ such that

$$\hat{\boldsymbol{\sigma}}_0 = \begin{pmatrix} \partial^2\mathfrak{A}/\partial z^2 & -\partial^2\mathfrak{A}/\partial y\,\partial z \\ -\partial^2\mathfrak{A}/\partial y\,\partial z & \partial^2\mathfrak{A}/\partial y^2 \end{pmatrix}, \tag{3.6}$$

so that (3.3a) is satisfied automatically. From (2.4), (2.5) and (3.3) we deduce that

$$\hat{\nabla}^2 p_0 = -\hat{\nabla}^2\, \partial u_0/\partial x = 0. \tag{3.7}$$

and from (2.4), (2.5) and (3.6) we then deduce that

$$\hat{\nabla}^4 \mathfrak{A} = 0. \tag{3.8}$$

From (3.4) we see that $\partial\mathfrak{A}/\partial y$ and $\partial\mathfrak{A}/\partial z$ are functions of x and t only on $G_0(x,y,z,t)=0$ and it follows that $\mathfrak{A}$ is linear in y and z, so that

$$\hat{\boldsymbol{\sigma}}_0 = \mathbf{0}, \tag{3.9}$$

identically. Hence σ_{xx0} is the only non-zero component of the leading-order stress tensor $\boldsymbol{\sigma}_0$.

Equations (2.4), (2.5) and (3.9) imply

$$\frac{\partial v_0}{\partial y} = \frac{p_0}{2}, \quad \frac{\partial w_0}{\partial z} = \frac{p_0}{2}, \quad \frac{\partial v_0}{\partial z} + \frac{\partial w_0}{\partial y} = 0, \tag{3.10a–c}$$

and from the continuity equation, (3.3b), and (3.10), we have

$$\frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = p_0 = -\frac{\partial u_0}{\partial x}(x,t). \tag{3.11a–c}$$

From (3.10) and (3.11) we then have

$$v_0 = -\frac{y}{2}\frac{\partial u_0}{\partial x} + a(x,t)\,z + b(x,t), \quad w_0 = -\frac{z}{2}\frac{\partial u_0}{\partial x} - a(x,t)\,y + c(x,t), \tag{3.12a, b}$$

where $a(x,t)$, $b(x,t)$ and $c(x,t)$ are also undetermined eigenfunctions. They arise because (3.3)–(3.4) is a two-dimensional zero-boundary-stress Stokes' flow problem and its solution is unique only up to an arbitrary rigid-body translation (b and c) and an arbitrary rigid-body rotation (a). Hence, even if we knew $u_0(x,t)$, the problem for the transverse leading-order velocities v_0 and w_0 would not be determinate. Nonetheless, we will be able to determine u_0, v_0 and w_0 by considering the $O(\epsilon^2)$ terms in (2.6) and (2.8), and we will do so in the following section.

First, however, we deduce a physically obvious mass conservation equation for the leading-order extensional flow, from the $O(1)$ equations, as follows. We need two forms of the Reynolds' transport theorem, namely that if, for x, t fixed, $\mathscr{A}$ is an area bounded by the closed simple curve $G(x,y,z,t)=0$ and $\phi(x,y,z,t)$ is any differentiable function on $\mathscr{A}$ then

$$\frac{\partial}{\partial t}\iint_{\mathscr{A}} \phi\,\mathrm{d}y\,\mathrm{d}z = \iint_{\mathscr{A}} \frac{\partial \phi}{\partial t}\mathrm{d}y\,\mathrm{d}z - \oint_{\partial\mathscr{A}} \frac{\partial G}{\partial t}\frac{\phi\,\mathrm{d}s}{|\hat{\boldsymbol{\nabla}} G|}, \tag{3.13a}$$

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}} \phi\,\mathrm{d}y\,\mathrm{d}z = \iint_{\mathscr{A}} \frac{\partial \phi}{\partial x}\mathrm{d}y\,\mathrm{d}z - \oint_{\partial\mathscr{A}} \frac{\partial G}{\partial x}\frac{\phi\,\mathrm{d}s}{|\hat{\boldsymbol{\nabla}} G|}, \tag{3.13b}$$

where $\partial\mathscr{A}$ denotes the curve $G(x,y,z,t)=0$.

If we integrate the leading-order kinematic condition (2.7) around the closed curve $G_0(x,y,z,t)=0$, we obtain

$$-\oint_{\partial\mathscr{A}_0}\left(\frac{\partial G_0}{\partial t}+u_0\frac{\partial G_0}{\partial x}\right)\frac{\mathrm{d}s}{|\hat{\nabla}G_0|}=\oint_{\partial\mathscr{A}_0}\left(v_0\frac{\partial G_0}{\partial y}+w_0\frac{\partial G_0}{\partial z}\right)\frac{\mathrm{d}s}{|\hat{\nabla}G_0|}$$
$$=\oint_{\partial\mathscr{A}_0}\hat{\boldsymbol{q}}_0\cdot\hat{\boldsymbol{n}}_0\,\mathrm{d}s$$
$$=\iint_{\mathscr{A}_0}\hat{\nabla}\cdot\hat{\boldsymbol{q}}_0\,\mathrm{d}y\,\mathrm{d}z,$$

where $\hat{\boldsymbol{n}}_0$ is the transverse normal to the curve (3.5). Then, from the leading-order continuity equation (3.3*b*) we find that

$$\iint_{\mathscr{A}_0}\frac{\partial u_0}{\partial x}\mathrm{d}y\,\mathrm{d}z-\oint_{\partial\mathscr{A}_0}\left(\frac{\partial G_0}{\partial t}+u_0\frac{\partial G_0}{\partial x}\right)\frac{\mathrm{d}s}{|\hat{\nabla}G_0|}=0,$$

and, by (3.13),

$$\frac{\partial}{\partial t}\iint_{\mathscr{A}_0}\mathrm{d}y\,\mathrm{d}z+\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}u_0(x,t)\,\mathrm{d}y\,\mathrm{d}z=0. \tag{3.14}$$

Putting $A_0=\iint_{\mathscr{A}_0}\mathrm{d}y\,\mathrm{d}z$, this becomes

$$\frac{\partial A_0}{\partial t}+\frac{\partial}{\partial x}(u_0A_0)=0. \tag{3.15}$$

4. The first-order equations

To determine u_0, v_0 and w_0 and the leading-order flow completely, we must proceed to the $O(\epsilon^2)$ terms in (2.6) and (2.8). The $O(\epsilon^2)$ terms from (2.6) can be written in the form

$$\hat{\nabla}\cdot\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix}=-\frac{\partial\sigma_{xx0}}{\partial x},\quad \hat{\nabla}\cdot\hat{\boldsymbol{\sigma}}_1=-\frac{\partial}{\partial x}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix},\quad \hat{\nabla}\cdot\hat{\boldsymbol{q}}_1=-\frac{\partial u_1}{\partial x}. \tag{4.1\,a–c}$$

The $O(\epsilon^2)$ terms in (2.8) simplify since the only non-zero component of $\boldsymbol{\sigma}_0$ is σ_{xx0}, and we find that they can be written as

$$\left.\begin{aligned}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix}\cdot\hat{\boldsymbol{n}}_0&=\frac{-1}{|\hat{\nabla}G_0|}\frac{\partial G_0}{\partial x}\sigma_{xx0} &(4.2a)\\ \hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{n}}_0&=\frac{-1}{|\hat{\nabla}G_0|}\frac{\partial G_0}{\partial x}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix} &(4.2b)\end{aligned}\right\}\quad\text{on}\quad G_0(x,y,z,t)=0.$$

We obtain an axial momentum equation by integrating (4.1*a*) over the cross-section $\mathscr{A}_0$ and applying the divergence theorem;

$$\iint_{\mathscr{A}_0}\frac{\partial\sigma_{xx0}}{\partial x}\mathrm{d}y\,\mathrm{d}z=-\oint_{\partial\mathscr{A}_0}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix}\cdot\hat{\boldsymbol{n}}_0\,\mathrm{d}s.$$

From (4.2*b*) and (3.13) we than obtain the constant-tension condition

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}\sigma_{xx0}\,\mathrm{d}y\,\mathrm{d}z=0. \tag{4.3}$$

Since $\sigma_{xx0} = -p_0 + 2(\partial u_0/\partial x) = 3(\partial u_0/\partial x)$, we can also write this as

$$\frac{\partial}{\partial x}\left(3A_0\frac{\partial u_0}{\partial x}\right) = 0. \tag{4.4}$$

Together, (3.15) and (4.4) constitute a hyperbolic system for u_0 and A_0. They require boundary and initial conditions, and we take these to be the initial leading-order area of the fibre,

$$A_0(x,0) = \iint_{\mathscr{A}_0|_{t=0}} \mathrm{d}y\,\mathrm{d}z,$$

and the axial velocity of the end planes of the fibre,

$$u_0(s_i(t),t) = \dot{s}(t) \quad (i = 1,2).$$

Equations (3.15) and (4.4) are equivalent to (1.1) and, together with their boundary and initial conditions, they completely determine the evolution of an axisymmetric fibre. For a fibre that is not axisymmetric, however, we need three more equations to eliminate the arbitrary rigid-body motions in (3.12), and to determine v_0 and w_0. We obtain these as follows.

Integrating (4.1*b*) over a cross-section $\mathscr{A}_0$ and applying the divergence theorem gives

$$\iint_{\mathscr{A}_0} \frac{\partial}{\partial x}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix}\mathrm{d}y\,\mathrm{d}z = -\oint_{\partial\mathscr{A}_0} \hat{\boldsymbol{\sigma}}_1\cdot\hat{\boldsymbol{n}}_0\,\mathrm{d}s$$

and from (4.2*b*) and (3.13) we obtain

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}\begin{pmatrix}\sigma_{xy1}\\ \sigma_{xz1}\end{pmatrix}\mathrm{d}y\,\mathrm{d}z = \mathbf{0}. \tag{4.5}$$

From (2.4) and (2.5) we have $\sigma_{xy1} = \partial u_1/\partial y + \partial v_0/\partial x$ and $\sigma_{xz1} = \partial u_1/\partial z + \partial \omega_0/\partial x$, so we can write this as

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}\left(\frac{\partial \hat{\boldsymbol{q}}_0}{\partial x} + \hat{\boldsymbol{\nabla}}u_1\right)\mathrm{d}y\,\mathrm{d}z = \mathbf{0}. \tag{4.6}$$

From (4.1*a*, (3.12) and (4.2*b*) we see that u_1 satisfies the boundary-value problem

$$\hat{\boldsymbol{\nabla}}^2 \mathrm{u}_1 = -2\frac{\partial^2 u_0}{\partial x^2}, \tag{4.7a}$$

$$\hat{\boldsymbol{\nabla}}u_1\cdot\hat{\boldsymbol{\nabla}}G_0 = -3\frac{\partial u_0}{\partial x}\frac{\partial G_0}{\partial x} - \left(\frac{\partial \hat{\boldsymbol{q}}_0}{\partial x}\right)\cdot\hat{\boldsymbol{\nabla}}G_0 \quad \text{on} \quad G_0(x,y,z,t) = 0. \tag{4.7b}$$

Note that the consistency condition for (4.7), namely

$$-2\iint_{\mathscr{A}_0}\frac{\partial^2 u_0}{\partial x^2}\mathrm{d}y\,\mathrm{d}z = -3\oint_{\partial\mathscr{A}_0}\frac{\partial u_0}{\partial x}\frac{\partial G_0}{\partial x}\mathrm{d}s\,|\hat{\boldsymbol{\nabla}}G_0| - \oint_{\partial\mathscr{A}_0}\frac{\partial \hat{\boldsymbol{q}}_0}{\partial x}\cdot\hat{\boldsymbol{n}}_0\,\mathrm{d}s,$$

reduces to (4.4) after using (3.12) to eliminate $\hat{\boldsymbol{q}}_0$, applying the divergence theorem to eliminate a, b and c, and then applying (3.13*b*). So, in principle, (4.7) determines u_1 up to an arbitrary additive function of x and t, as long as we assume that $u_0(x,t)$ is determined from (3.15) and (4.4) and that $\hat{\boldsymbol{q}}_0$ is known or, equivalently, that a, b and c in (3.12) are known; this knowledge would then allow us to find G_0 from (2.7).

We can use (4.6) and (4.7) to determine the evolution of the centreline of the fibre. To do so we need the identities

$$\iint_{\mathscr{A}_0} \frac{\partial \phi}{\partial y}\,\mathrm{d}y\,\mathrm{d}z = \oint_{\partial\mathscr{A}_0} \phi \frac{\partial G_0}{\partial y}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|} = \oint_{\partial\mathscr{A}_0} y\frac{\partial \phi}{\partial \hat{n}}\,\mathrm{d}s - \iint_{\mathscr{A}_0} y\hat{\nabla}^2\phi\,\mathrm{d}y\,\mathrm{d}z, \tag{4.8a}$$

$$\iint_{\mathscr{A}_0} \frac{\partial \phi}{\partial z}\,\mathrm{d}y\,\mathrm{d}z = \oint_{\partial\mathscr{A}_0} \phi \frac{\partial G_0}{\partial z}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|} = \oint_{\partial\mathscr{A}_0} y\frac{\partial \phi}{\partial \hat{n}}\,\mathrm{d}s - \iint_{\mathscr{A}_0} z\hat{\nabla}^2\phi\,\mathrm{d}y\,\mathrm{d}z, \tag{4.8b}$$

where $\phi(x, y, z, t)$ is any twice continuously differentiable function. These follow from the divergence theorem and the vector identity

$$\hat{\nabla}^2\left(\begin{pmatrix} y \\ z \end{pmatrix}\phi\right) = \begin{pmatrix} y \\ z \end{pmatrix}\hat{\nabla}^2\phi + 2\hat{\nabla}\phi.$$

Consider, for example, the first component of the double integral in (4.6),

$$I_1 = \iint_{\mathscr{A}_0}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_1}{\partial y}\right)\mathrm{d}y\,\mathrm{d}z.$$

From (4.8a) we have

$$I_1 = \oint_{\partial\mathscr{A}_0} y\frac{\partial u_1}{\partial \hat{n}}\,\mathrm{d}s + \iint_{\mathscr{A}_0}\left(\frac{\partial v_0}{\partial x} - y\hat{\nabla}^2 u_1\right)\mathrm{d}y\,\mathrm{d}z,$$

and, using (4.7) to eliminate $\partial u_1/\partial \hat{n}_0$ and $\hat{\nabla}^2 u_1$, and (3.12) to eliminate $\partial v_0/\partial x$ and $\partial w_0/\partial x$, we find that

$$\begin{aligned} I_1 = {} & \frac{\partial^2 u_0}{\partial x^2}\left(\frac{3}{2}\iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{2}\oint_{\partial\mathscr{A}_0}\left(y^2\frac{\partial G_0}{\partial y} + yz\frac{\partial G_0}{\partial z}\right)\frac{\mathrm{d}s}{|\hat{\nabla} G_0|}\right) - 3\frac{\partial u_0}{\partial x}\oint_{\partial\mathscr{A}_0} y\frac{\partial G_0}{\partial x}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|} \\ & + \frac{\partial a}{\partial x}\left(\iint_{\mathscr{A}_0} z\,\mathrm{d}y\,\mathrm{d}z - \oint_{\partial\mathscr{A}_0}\left(yz\frac{\partial G_0}{\partial y} - y^2\frac{\partial G_0}{\partial z}\right)\frac{\mathrm{d}s}{|\hat{\nabla} G_0|}\right) \\ & + \frac{\partial b}{\partial x}\left(\iint_{\mathscr{A}_0}\mathrm{d}y\,\mathrm{d}z - \oint_{\partial\mathscr{A}_0} y\frac{\partial G_0}{\partial y}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|}\right) - \frac{\partial c}{\partial x}\oint_{\partial\mathscr{A}_0} y\frac{\partial G_0}{\partial z}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|}. \end{aligned}$$

Using (4.8) to simplify the above, we find that the coefficients of $\partial a/\partial x$, $\partial b/\partial x$ and $\partial c/\partial x$ all vanish identically and that

$$I_1 = 3\left(\frac{\partial^2 u_0}{\partial x^2}\iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial u_0}{\partial x}\oint_{\partial\mathscr{A}_0} y\frac{\partial G_0}{\partial x}\frac{\mathrm{d}s}{|\hat{\nabla} G_0|}\right) = 3\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x}\iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z\right),$$

after applying the Reynolds' transport theorem, (3.13). A similar calculation for the second component of the double integral in (4.6) gives

$$\iint_{\mathscr{A}_0}\left(\frac{\partial w_0}{\partial x} + \frac{\partial u_1}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z = 3\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x}\iint_{\mathscr{A}_0} z\,\mathrm{d}y\,\mathrm{d}z\right).$$

Thus, (4.6) implies that

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_0}{\partial x}\iint_{\mathscr{A}_0}\begin{pmatrix} y \\ z \end{pmatrix}\mathrm{d}y\,\mathrm{d}z\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.9}$$

or, equivalently that

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{A_0(x,t)}\iint_{\mathscr{A}_0}\begin{pmatrix} y \\ z \end{pmatrix}\mathrm{d}y\,\mathrm{d}z\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This then implies that the centreline of the fibre (y_c, z_c), defined by

$$\begin{pmatrix} y_c(x,t) \\ z_c(x,t) \end{pmatrix} = \frac{1}{A_0(x,t)} \iint_{\mathscr{A}_0} \begin{pmatrix} y \\ z \end{pmatrix} \mathrm{d}y\,\mathrm{d}z,$$

lies along the straight line joining the centres of the two ends. We recall from the start of §2 that this line can move arbitrarily yet leave our model invariant. We take this line to be the x-axis, so that

$$\iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z = \iint_{\mathscr{A}_0} z\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{4.10}$$

If the centreline is initially straight, it remains straight for all $t > 0$ in our model. If it is not initially straight, it will straighten out on a timescale small compared to our timescale L/U (see Buckmaster & Nachman 1978; Buckmaster *et al.* 1975; Wilmott 1989); in this case our choice of timescale L/U is clearly inappropriate.

To find b and c, we proceed as follows. Differentiating (4.10) with respect to t and x, using the leading-order continuity condition (2.7), and (3.13), gives

$$0 = \frac{\partial}{\partial t} \iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z = \oint_{\partial\mathscr{A}_0} \left(u_0 \frac{\partial G_0}{\partial x} + v_0 \frac{\partial G_0}{\partial y} + w_0 \frac{\partial G_0}{\partial z} \right) \frac{y}{|\hat{\boldsymbol{\nabla}} G_0|} \mathrm{d}s,$$

$$0 = \frac{\partial}{\partial x} \iint_{\mathscr{A}_0} y\,\mathrm{d}y\,\mathrm{d}z = -\oint_{\partial\mathscr{A}_0} \frac{\partial G_0}{\partial x} \frac{y}{|\hat{\boldsymbol{\nabla}} G_0|} \mathrm{d}s.$$

From (3.12), we then obtain

$$\oint_{\partial\mathscr{A}_0} \left(\left(-\frac{y}{2}\frac{\partial u_0}{\partial x} + az + b \right) \frac{\partial G_0}{\partial y} + \left(-\frac{z}{2}\frac{\partial u_0}{\partial x} - ay + c \right) \frac{\partial G_0}{\partial z} \right) \frac{y}{|\hat{\boldsymbol{\nabla}} G_0|} \mathrm{d}s = 0,$$

so that using the divergence theorem we get

$$0 = \iint_{\mathscr{A}_0} \left(-\frac{3}{2}\frac{\partial u_0}{\partial x} y + az + b \right) \mathrm{d}y\,\mathrm{d}z = bA_0,$$

since $\partial u_0/\partial x$, $a(x,t)$ and $b(x,t)$ are independent of y and z. A similar argument shows that $c(x,t) = 0$.

The third and final equation can be obtained by taking moments of (4.1b). From the divergence theorem and the symmetry of $\hat{\boldsymbol{\sigma}}$ we find that

$$-\iint_{\mathscr{A}_0} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \sigma_{xy1} \\ \sigma_{xz1} \end{pmatrix} \mathrm{d}y\,\mathrm{d}z = \iint_{\mathscr{A}_0} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \wedge (\hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{\sigma}})\,\mathrm{d}y\,\mathrm{d}z$$

$$= \oint_{\partial\mathscr{A}_0} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \wedge (\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{n}}_0)\,\mathrm{d}s$$

and, from (4.2) and (3.13), we deduce that

$$\frac{\partial}{\partial x} \iint_{\mathscr{A}_0} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} 0 \\ \sigma_{xy1} \\ \sigma_{xz1} \end{pmatrix} \mathrm{d}y\,\mathrm{d}z = \mathbf{0}, \tag{4.11}$$

or, equivalently, that

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}\left(y\frac{\partial u_1}{\partial z}-z\frac{\partial u_1}{\partial y}+y\frac{\partial w_0}{\partial x}-z\frac{\partial v_0}{\partial x}\right)\mathrm{d}y\,\mathrm{d}z=0. \tag{4.12}$$

Using the fact that $b = c = 0$ in (3.12), (4.12) becomes

$$\frac{\partial}{\partial x}\iint_{\mathscr{A}_0}\left(y\frac{\partial u_1}{\partial z}-z\frac{\partial u_1}{\partial y}-(y^2+z^2)\frac{\partial a}{\partial x}\right)\mathrm{d}y\,\mathrm{d}z=0. \tag{4.13}$$

Since (4.7) determines u_1 up to an arbitrary additive function of x and t if u_0 and a are known, (4.13) allows us to determine $a(x,t)$ in principle.

In summary, assuming that we know u_0 and A_0 from (3.15) and (4.4), we now have a closed system of equations for the leading-order shape G_0 and rotation a. A solution algorithm might be to time-step for G_0 as follows. Using known initial values of a, and hence $\hat{\boldsymbol{q}}_0$, and u_1, solve (2.7) for an updated G_0. Then, use the known $\hat{\boldsymbol{q}}_0$ and the updated G_0 to find an updated u_1 from the boundary-value problem (4.7). Finally, use the updated u_1 and G_0 to update a and hence $\hat{\boldsymbol{q}}_0$ from (4.13). This final step will require the application of boundary condition, such as a specified twist at each end of the fibre.

In the next section we will show how to reduce this cumbersome solution procedure for u_1 to a once-and-for-all boundary-value problem by using appropriately modified Lagrangian variables.

5. Lagrangian description

We begin by introducing new variables ξ, η and ζ that are Lagrangian variables for the leading-order velocity field (u_0, v_0, w_0), modified to account for cross-sectional area variations. We set

$$x = X(\xi,\tau), \tag{5.1a}$$

$$y = Y(\xi,\eta,\zeta,\tau) = A(\xi,\tau)^{\frac{1}{2}}(\eta\cos\theta+\zeta\sin\theta), \tag{5.1b}$$

$$z = Z(\xi,\eta,\zeta,\tau) = A(\xi,\tau)^{\frac{1}{2}}(\zeta\cos\theta-\eta\sin\theta), \tag{5.1c}$$

$$t = \tau, \tag{5.1d}$$

where
$$A(\xi,\tau) = A_0(x,t), \tag{5.2}$$

and θ, X, Y and Z are defined by

$$\frac{\partial\theta}{\partial\tau} = a(X(\xi,\tau),\tau) = \alpha(\xi,\tau), \quad \theta(0) = 0, \text{ say}, \tag{5.3a}$$

$$\frac{\partial X}{\partial\tau} = u_0(X(\xi,\tau),\tau), \quad X(\xi,0) = \xi, \tag{5.3b}$$

$$\frac{\partial Y}{\partial\tau} = v_0(X(\xi,\tau),Y(\xi,\eta,\zeta,\tau),Z(\xi,\eta,\zeta,\tau),\tau), \quad Y(\xi,\eta,\zeta,0) = A(\xi,0)^{\frac{1}{2}}\eta, \tag{5.3c}$$

$$\frac{\partial Z}{\partial\tau} = w_0(X(\xi,\tau),Y(\xi,\eta,\zeta,\tau),Z(\xi,\eta,\zeta,\tau),\tau), \quad Z(\xi,\eta,\zeta,0) = A(\xi,0)^{\frac{1}{2}}\zeta. \tag{5.3d}$$

Thus from (3.15),

$$\frac{\partial A}{\partial\tau}+\frac{\partial u_0}{\partial x}A = 0. \tag{5.4}$$

We also note that (4.4) implies that

$$\frac{\partial}{\partial \xi}\left(\frac{A}{\partial X/\partial \xi}\frac{\partial u_0}{\partial \xi}\right) = 0. \tag{5.5}$$

Setting $\mathscr{G} = 0$ to be any Lagrangian description of the leading-order fibre surface, the kinematic condition (2.7) implies

$$\frac{\partial \mathscr{G}}{\partial \tau} = 0. \tag{5.6}$$

Thus, we see from (5.3) that $\mathscr{G}$ can be chosen to be

$$G_0(x, y, z, 0) = \mathscr{G}(\xi, \eta, \zeta).$$

The fact that $\mathscr{G}$ is a known function of our special choice of Lagrangian variables reflects the fact that each cross-section retains its shape but may both scale affinely and rotate with respect to its centroid.

We now consider our equations (4.7) and (4.13) for u_1 and a in these new variables. We need the chain rule identities

$$\begin{aligned}\frac{\partial G_0}{\partial t} = \frac{\partial \mathscr{G}}{\partial \tau} - \frac{\partial X/\partial \tau}{\partial X/\partial \xi}\left(\frac{\partial \mathscr{G}}{\partial \xi} - \frac{\partial A/\partial \xi}{2A}\left(\eta\frac{\partial \mathscr{G}}{\partial \eta} + \zeta\frac{\partial \mathscr{G}}{\partial \zeta}\right) - \frac{\partial \theta}{\partial \xi}\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta} - \eta\frac{\partial \mathscr{G}}{\partial \zeta}\right)\right)\\ - \frac{\partial A/\partial \tau}{2A}\left(\eta\frac{\partial \mathscr{G}}{\partial \eta} + \zeta\frac{\partial \mathscr{G}}{\partial \zeta}\right) - \frac{\partial \theta}{\partial \tau}\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta} - \eta\frac{\partial \mathscr{G}}{\partial \zeta}\right),\end{aligned} \tag{5.7a}$$

$$\frac{\partial G_0}{\partial x} = \frac{1}{\partial X/\partial \xi}\left(\frac{\partial \mathscr{G}}{\partial \xi} - \frac{\partial A/\partial \xi}{2A}\left(\eta\frac{\partial \mathscr{G}}{\partial \eta} + \zeta\frac{\partial \mathscr{G}}{\partial \zeta}\right) - \frac{\partial \theta}{\partial \xi}\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta} - \eta\frac{\partial \mathscr{G}}{\partial \zeta}\right)\right), \tag{5.7b}$$

$$\frac{\partial G_0}{\partial y} = \frac{1}{(A(\xi, \tau))^{\frac{1}{2}}}\left(\cos\theta\frac{\partial \mathscr{G}}{\partial \eta} + \sin\theta\frac{\partial \mathscr{G}}{\partial \zeta}\right), \tag{5.7c}$$

$$\frac{\partial G_0}{\partial z} = \frac{1}{(A(\xi, \tau))^{\frac{1}{2}}}\left(\cos\theta\frac{\partial \mathscr{G}}{\partial \zeta} - \sin\theta\frac{\partial \mathscr{G}}{\partial \eta}\right). \tag{5.7d}$$

In order to simplify (4.7) as much as possible, we write

$$u_1 = A(\xi, \tau)\left(\tilde{u}(\xi, \tau) - \tfrac{1}{2}(\eta^2 + \zeta^2)\frac{\partial^2 u_0}{\partial x^2}\right); \tag{5.8}$$

we find that $\tilde{u}$ satisfies the Neumann boundary value problem

$$\tilde{\nabla}^2\tilde{u} = 0, \tag{5.9a}$$

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial \eta}\frac{\partial \mathscr{G}}{\partial \eta} + \frac{\partial \tilde{u}}{\partial \zeta}\frac{\partial \mathscr{G}}{\partial \zeta} = \\ -\left[\frac{3}{(\partial X/\partial \xi)^2}\frac{\partial u_0}{\partial \xi}\frac{\partial \mathscr{G}}{\partial \xi} + \left(\frac{3}{(\partial X/\partial \xi)^2}\frac{\partial u_0}{\partial \xi}\frac{\partial \theta}{\partial \xi} - \frac{1}{\partial X/\partial \xi}\frac{\partial \alpha}{\partial \xi}\right)\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta} - \eta\frac{\partial \mathscr{G}}{\partial \zeta}\right)\right], \quad \text{on} \quad \mathscr{G} = 0,\end{aligned} \tag{5.9b}$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}.$$

We can further simplify the dependence of $\tilde{u}$ on α and θ by noting that the transport equations (3.12) become

$$\frac{\partial A}{\partial \tau}\iint_{\mathscr{A}} \phi \,\mathrm{d}\eta\,\mathrm{d}\zeta+\oint_{\partial\mathscr{A}}\left(\frac{1}{2}\frac{\partial A}{\partial \tau}\left(\eta\frac{\partial \mathscr{G}}{\partial \eta}+\zeta\frac{\partial \mathscr{G}}{\partial \zeta}\right)-A\frac{\partial \theta}{\partial \tau}\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta}-\eta\frac{\partial \mathscr{G}}{\partial \zeta}\right)\right)\frac{\mathrm{d}s}{|\tilde{\nabla}\mathscr{G}|}=0, \quad (5.10a)$$

$$\frac{\partial A}{\partial \xi}\iint_{\mathscr{A}} \phi \,\mathrm{d}\eta\,\mathrm{d}\zeta+\oint_{\partial\mathscr{A}}\left(A\frac{\partial \mathscr{G}}{\partial \xi}-\frac{1}{2}\frac{\partial A}{\partial \xi}\left(\eta\frac{\partial \mathscr{G}}{\partial \eta}+\zeta\frac{\partial \mathscr{G}}{\partial \zeta}\right)-A\frac{\partial \theta}{\partial \xi}\left(\zeta\frac{\partial \mathscr{G}}{\partial \eta}-\eta\frac{\partial \mathscr{G}}{\partial \zeta}\right)\right)\frac{\mathrm{d}s}{|\tilde{\nabla}\mathscr{G}|}=0, \quad (5.10b)$$

where $\tilde{\nabla}=(\partial/\partial\eta,\partial/\partial\zeta)$. Hence, if we put $\phi=1$ in (4.8) and apply the divergence theorem,

$$\oint_{\partial\mathscr{A}}\frac{\partial \mathscr{G}}{\partial \xi}\frac{\mathrm{d}s}{|\tilde{\nabla}\mathscr{G}|}=0. \quad (5.11)$$

This means that we can write $\tilde{u}$ as a combination of two functions that are independent of τ;

$$\tilde{u}=\frac{-3}{(\partial X/\partial \xi)^2}\frac{\partial u_0}{\partial \xi}\tilde{u}_1-\left(\frac{3}{(\partial X/\partial \xi)^2}\frac{\partial u_0}{\partial \xi}\frac{\partial \theta}{\partial \xi}-\frac{1}{\partial X/\partial \xi}\frac{\partial \alpha}{\partial \xi}\right)\tilde{u}_2, \quad (5.12)$$

where $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the Neumann problems

$$\nabla^2\tilde{u}_1=0,\quad \frac{\partial \tilde{u}_1}{\partial \eta}\frac{\partial \mathscr{G}}{\partial \eta}+\frac{\partial \tilde{u}_1}{\partial \zeta}\frac{\partial \mathscr{G}}{\partial \zeta}=\frac{\partial \mathscr{G}}{\partial \xi}\quad \text{on}\quad \mathscr{G}(\xi,\eta,\zeta)=0, \quad (5.13)$$

and

$$\tilde{\nabla}^2\tilde{u}_2=0,\quad \frac{\partial \tilde{u}_2}{\partial \eta}\frac{\partial \mathscr{G}}{\partial \eta}+\frac{\partial \tilde{u}_2}{\partial \zeta}\frac{\partial \mathscr{G}}{\partial \zeta}=\zeta\frac{\partial \mathscr{G}}{\partial \eta}-\eta\frac{\partial \mathscr{G}}{\partial \zeta}\quad \text{on}\quad \mathscr{G}(\xi,\eta,\zeta)=0. \quad (5.14)$$

Finally, the evolution of α and θ are given by $\partial\theta/\partial\tau=\alpha$ together with the Lagrangian form of (4.13), namely

$$\frac{\partial}{\partial \xi}\left\{3A^2\left(\frac{\partial u_0/\partial \xi}{(\partial X/\partial \xi)^2}\frac{\partial \theta}{\partial \xi}-\frac{1}{\partial X/\partial \xi}\frac{\partial^2\theta}{\partial \tau\,\partial \xi}\right)\iint_{A_{00}}\left(\eta\frac{\partial \tilde{u}_2}{\partial \zeta}-\zeta\frac{\partial \tilde{u}_2}{\partial \eta}\right)\mathrm{d}\eta\,\mathrm{d}\zeta\right.$$
$$\left.+3A^2\frac{\partial u_0/\partial \xi}{(\partial X/\partial \xi)^2}\iint_{A_{00}}\left(\eta\frac{\partial \tilde{u}_1}{\partial \zeta}-\zeta\frac{\partial \tilde{u}_1}{\partial \eta}\right)\mathrm{d}\eta\,\mathrm{d}\zeta+\frac{A^2}{\partial X/\partial \xi}\frac{\partial^2\theta}{\partial \xi\,\partial \tau}\iint_{A_{00}}(\eta^2+\zeta^2)\,\mathrm{d}\eta\,\mathrm{d}\zeta\right\}=0, \quad (5.15)$$

where A_{00} is the cross-sectional area at $t=0$. The highest derivative of θ in (5.15) is $\partial^3\theta/\partial^2\xi\,\partial\tau$, and so (5.15) is the ξ-derivative of a wave equation for θ. Suitable boundary and initial conditions for θ are

$$\theta(\xi,0)=0,\quad \theta(0,\tau)=0,\quad \theta(1,\tau)=\Omega(\tau). \quad (5.16)$$

Note that the stretching and twisting are uncoupled since u_0 and A_0 and hence the amount of stretch are determined through (3.15) and (4.4), and then the twist is given by (5.15). It is shown in Dewynne *et al.* (1989) that the solution of (3.15) and (4.4) is independent of the history of the stretching. This, however, is not generally true of solutions of (5.15); that is, the stretch will be history independent, but the twist will in general depend on the history of the deformation. Also, observe that the inhomogeneous term in (5.15) is proportional to

$$\iint_{A_{00}}\left(\eta\frac{\partial \tilde{u}_1}{\partial \zeta}-\zeta\frac{\partial \tilde{u}_1}{\partial \eta}\right)\mathrm{d}\eta\,\mathrm{d}\zeta.$$

If this quantity is non-zero, there will be twisting of the fibre when the ends are pulled apart even with the zero boundary conditions (5.16).

6. Examples – fibres with elliptical cross-sections

We start with a fibre whose surface is initially given by

$$\left(\frac{y}{\beta(x)}\right)^2+\left(\frac{z}{\gamma(x)}\right)^2-1=0, \tag{6.1}$$

where $\beta(x)$ and $\gamma(x)$ are given positive functions. Our Lagrangian description of the surface is

$$\mathscr{G}(\xi,\eta,\zeta)=A(\xi,0)\left[\left(\frac{\eta}{\beta(\xi)}\right)^2+\left(\frac{\zeta}{\gamma(\xi)}\right)^2\right]-1=0$$

at any subsequent time. Since $A(\xi,0)=\pi\beta(\xi)\gamma(\xi)$, we can explicitly solve (5.13), (5.14) for $\tilde{u}_1$ and $\tilde{u}_2$, and we find

$$\tilde{u}_1=\frac{1}{4}\left(\frac{\gamma'(\xi)}{\gamma(\xi)}-\frac{\beta'(\xi)}{\beta(\xi)}\right)(\eta^2-\zeta^2),\quad \tilde{u}_2=\left(\frac{\gamma^2-\beta^2}{\gamma^2+\beta^2}\right)\eta\zeta. \tag{6.2a,b}$$

In order to simplify (5.15), we note the following:

$$\iint_{A_{00}}\left(\eta\frac{\partial\tilde{u}_2}{\partial\zeta}-\zeta\frac{\partial\tilde{u}_2}{\partial\eta}\right)\mathrm{d}\eta\,\mathrm{d}\zeta=\frac{-\pi(\beta^2-\gamma^2)^2}{4\gamma\beta(\gamma^2+\beta^2)},$$

$$\iint_{A_{00}}\left(\eta\frac{\partial\tilde{u}_1}{\partial\zeta}-\zeta\frac{\partial\tilde{u}_1}{\partial\eta}\right)\mathrm{d}\eta\,\mathrm{d}\zeta=0,$$

$$\iint_{A_{00}}(\eta^2+\zeta^2)\,\mathrm{d}\eta\,\mathrm{d}\zeta=\frac{\pi}{4\gamma\beta}(\gamma^2+\beta^2).$$

(i) As a first specific example, consider the case of twist in the absence of stretching. In this case $A_0(x,t)=A(\xi,0)=\pi\beta(\xi)\gamma(\xi)$, $u_0=0$ and $X=\xi$. The non-dimensional length of the fibre is always unity in this case and therefore (5.15) becomes

$$\frac{\partial}{\partial\xi}\left(\beta\gamma\left(\frac{\beta^4+\gamma^4}{\beta^2+\gamma^2}\right)\frac{\partial^2\theta}{\partial\xi\,\partial\tau}\right)=0. \tag{6.3}$$

The solution of (6.3) with the boundary conditions (5.16) is

$$a(x,t)=\dot{\Omega}(t)\int_0^x f(\xi)\,\mathrm{d}\xi\bigg/\int_0^1 f(\xi)\,\mathrm{d}\xi, \tag{6.4a}$$

$$\theta(x,t)=\Omega(t)\int_0^x f(\xi)\,\mathrm{d}\xi\bigg/\int_0^1 f(\xi)\,\mathrm{d}\xi, \tag{6.4b}$$

where $f(\xi)=(\beta^2+\gamma^2)/(\beta\gamma(\beta^4+\gamma^4))$.

(ii) As a second example, consider the case of stretching and twisting of a fibre whose cross-section is initially elliptical and has uniform cross-sectional area:

$$A_0(x,0)=1=\pi\beta(x)\gamma(x).$$

Assuming that $u_0(0,t) = 0$ and $s(0) = 1$, the solution of (3.15) and (4.4) is

$$u_0(x,t) = \frac{\dot{s}(t)}{s(t)}x, \quad A_0(x,t) = \frac{1}{s(t)}.$$

Our Lagrangian description of the free surface is

$$\mathscr{G}(\xi,\eta,\zeta) = \left(\frac{\eta}{\beta(\xi)}\right)^2 + \left(\frac{\zeta}{\gamma(\xi)}\right)^2,$$

and, further, from (5.4), we clearly have

$$X(\xi,\tau) = s(\tau)\,\xi.$$

Equations (6.2) remain valid for $\tilde{u}_1$ and $\tilde{u}_2$, and we can write (5.15) for $\theta(\xi,\tau)$ as

$$\frac{\partial}{\partial\xi}\left\{-3\frac{\dot{s}}{s}\frac{(\beta^2-\gamma^2)^2}{\beta^2+\gamma^2}\frac{\partial\theta}{\partial\xi}+\frac{(\beta^4+\gamma^4)}{\beta^2+\gamma^2}\frac{\partial^2\theta}{\partial\tau\,\partial\xi}\right\} = 0,$$

where $\alpha = \partial\theta/\partial\tau$, and with boundary and initial conditions (5.16). For this special case, note that if $(\dot{\Omega}s)/(\Omega)$ is constant, then

$$\alpha = \dot{\Omega}(\tau)\,\tilde{\alpha}(\xi), \quad \theta = \Omega(\tau)\,\tilde{\theta}(\xi).$$

In the first and second cases above if $(\dot{\Omega}s)/(\Omega\dot{s})$ is constant, and in the case of stretching with no twist (see Dewynne *et al.* 1989), the fibre shape is independent of the history of the motion; in none of these cases does it matter how the fibre is stretched, only how far it has been stretched. In general, however, the twist will depend on the history of the stretching through (5.15). This may easily be seen by considering the two cases in which the fibre is first pulled and then twisted and *vice versa*. In the former case the amount of twist will be concentrated near the point of minimum cross-sectional area. In the latter case, the twist will be more evenly distributed. (Of course, the A and u_0 equations are independent of the history of the motion.) To see this, consider the simplest case of an axisymmetric fibre so that the twist equation (5.15) becomes

$$\frac{\partial}{\partial\xi}\left(\frac{A^2}{\partial X/\partial\xi}\frac{\partial^2\theta}{\partial\xi\,\partial\tau}\right) = 0,$$

or, returning to Eulerian variables,

$$\frac{\partial}{\partial x}\left(A^2\frac{\partial^2\theta}{\partial x\,\partial t}\right) = 0. \tag{6.5}$$

Thus

$$\frac{\partial\theta}{\partial x} = \int_0^t \frac{f(t')}{[A(x,t')]^2}\,\mathrm{d}t'$$

and the twist gradient can be seen to be concentrated where $A(x,t)$ is a minimum; this is exaggerated if the twisting takes place after the stretching.

7. Conclusion

We have presented a systematic method for calculating the extension and twist of a fibre of small aspect ratio ϵ. As in the theory of rods in elasticity, the information needed to close the equations for the leading-order motion is contained in the equations for the first-order motion. Given an initial fibre shape and drive mechanism, the procedure is to solve (3.15) and (4.4) as one would for an axisymmetric fibre, and also the elliptic boundary-value problems (5.13) and (5.14). The subsidiary functions X, θ and α, and hence the twist, are found from (5.3), (5.4) and (5.15) together with their initial and boundary conditions. This effectively means solving a wave equation whose solution appears to be history independent only in certain special cases.

We emphasize that we have discussed the response only on the timescale L/U; as mentioned after (4.10) a new model, involving more time derivatives, is needed to describe shorter time-scale behaviour. Indeed, the models proposed in Buckmaster & Nachman (1978), Buckmaster *et al.* (1975) and Wilmott (1989) do not have time reversibility properties similar to those of (1.1) and hence illustrate a clear distinction between, say, contracting and expanding fibres on these short timescales.

In seems unfortunate that the derivation of the principal new result, (5.15), takes so long when (1.1) and (4.10) can be written down almost at once if some very plausible physical assumptions are made. It seems, however, essential to have a systematic derivation of the model over the relatively simple time- and lengthscales assumed in this paper if further progress is to be made, say, for short-time motion or for two-dimensional flows in sheets.

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REFERENCES

BECHTEL, S. E., FOREST, M. G., HOLM, D. D. & LIN, K. J. 1988 *J. Fluid Mech.* **196**, 241.

BERIS, A. N. & LIU, B. 1988 *J. Non-Newtonian Fluid Mech.* **26**, 341.

BUCKMASTER, J. D. & NACHMAN, A. 1978 *Q. J. Mech. Appl. Maths* **31**, 157.

BUCKMASTER, J. D., NACHMAN, A. & TING, L. 1975 *J. Fluid Mech.* **69**, 1.

DEWYNNE, J. N., OCKENDON, J. R. & WILMOTT, P. 1989 *SIAM J. Appl. Maths* **49**, 983.

ENTOV, V. M. & YARIN, A. L. 1984 *J. Fluid. Mech.* **140**, 91.

GEYLING, F. T. 1976 *Bell Sys. Tech. J.* **55**, 1011.

GEYLING, F. T. & HOMSY, G. M. 1980 *Glass Technol.* **21**, 95.

KASÉ, S. 1974 *J. Appl. Polymer Sci.* **18**, 3267.

LOVE, A. E. H. 1927 *A Treatise on the Mathematical Theory of Elasticity*, 4th edn. Cambridge University Press.

MYERS, M. R. 1989 *AIChE J.* **35**, 592.

PARKER, D. F. 1984 *Z. Angew. Math. Phys.* **35**, 833.

PEARSON, J. R. A. & MATOVICH, M. A. 1969 *Indust. Engrg Chem. Fundam.* **8**, 605.

SCHULTZ, W. W. & DAVIS, S. H. 1982 *J. Rheol.* **26**, 331.

SHAH, F. T. & PEARSON, J. R. A. 1972 *Indust. Engrg Chem. Fundam.* **11**, 145.

WILMOTT, P. 1989 *Phys. Fluids* A **7**, 1098.